Time Reversal Symmetry and Humpty-Dumpty *

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It is argued that the quantal evolution of a physical system is fundamentally irreversible although the dynamical equations are symmetric under time reversal. The example of spin coherence in a Stern-Gerlach interferometer demonstrates that even undoing the evolution partially only may require sub-microscopic precision in controlling a macroscopic apparatus.

1. Elementary Considerations

In simplest nonrelativistic quantum mechanics – a particle with mass m moving along the x-axis under the influence of a potential energy $V(x)$ – the dynamics is governed by the Hamilton operator

$$\mathbf{H} = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}) > 0\,,$$

where, for simplicity, the requirement that $\mathbf{H}$ be bounded from below is equivalently replaced by insisting on the positivity of $\mathbf{H}$. The corresponding Schrödinger differential operator Ω,

$$\hbar\Omega = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\,,$$

appears in the Schrödinger equation for the spatial probability amplitude $\psi(t, x)$,

$$\left(\frac{\partial}{\partial t} + i\Omega\right)\psi(t, x) = 0\,.$$

It is well known and easily demonstrated that the time reversed wave function $(\Theta_T\psi)(t, x) = \psi^*(2T - t, x)$ is also a solution to this Schrödinger equation,

$$\left(\frac{\partial}{\partial t} + i\Omega\right)\psi^*(2T - t, x) = 0\,.$$

The anti-unitary time reversal operator Θ_T has these effects on spatial and momental wave functions:

$$(\Theta_T\psi)(T, x) = \psi^*(T, x)\,,$$
$$(\Theta_T\psi)(T, p) = \psi^*(T, -p)\,.$$

Perhaps most transparent is the transformation of the Wigner function $\varrho_w(t, x, p)$ associated with the state of the system: $(\Theta_T\varrho_w)(T, x, p) = \varrho_w(T, x, -p)$, signifying that the momentum, here identical with the velocity, is reversed. Indeed, for more complicated dynamics it is the velocities, not canonical momenta, that are reversed.

2. Undoing Previous Evolution

With the time reversal operator at hand one can imagine the undoing of all evolutionary changes that have occurred between, say, $t = 0$ and $t = T$ by first applying Θ_T, followed by evolution under the same Hamilton operator for $T < t < 2T$, and final application of Θ_{2T}, so that an instant τ after $t = 2T$ one has the same wave function as at time $t = \tau$:

$$\psi(2T + \tau, x) = e^{-i\Omega\tau}\Theta_{2T}e^{-i\Omega\tau}\Theta_T e^{-i\Omega\tau}\psi(0, x)$$
$$= \psi(\tau, x) = e^{-i\Omega\tau}\psi(0, x)\,.$$

This, however, cannot be achieved in any real sense. The changes symbolized by Θ_T and Θ_{2T} must be of dynamical origin themselves, which they cannot be because these operators are not unitary. It is true that all that really matters is the product

$$\Theta_{2T}e^{-i\Omega T}\Theta_T = e^{i\Omega T}$$

and this is unitary. But for most Ω the right-hand side is *not* of the form $\exp(-i\tilde{\Omega}T)$ with a physical, posi-

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tive $\tilde{\Omega}$. We are thus led to the conclusion that the undoing of all evolutionary changes is impossible. In other words: Quantum mechanical evolution is fundamentally irreversible.

3. The Humpty-Dumpty Problem

A more general scheme, not so tightly bound to elementary nonrelativistic quantum mechanics is what I would like to call the *Humpty-Dumpty problem.** Given an initial state with density operator $\varrho(0)$ and a Hamilton operator $H_1(t) > 0$, acting from $t = 0$ to $t = T$, find $\tilde{T}$ and $H_2(t) > 0$, acting from $t = T$ to $t = \tilde{T}$, such that the unitary evolution operators $U(0, T)$ and $U(T, \tilde{T})$, constructed as time ordered exponentials from $H_1(t)$ and $H_2(t)$, respectively, are effectively equal to the unit operator

$$U(0, T)\, U(T, \tilde{T}) \triangleq 1$$

when applied to the given $\varrho(0)$. The realization requires

$$\exp\Bigg[+\frac{i}{\hbar}\underbrace{\int_T^{\tilde{T}} H_2(t)\,dt}_{>0}\Bigg]_< \triangleq \exp\Bigg[-\frac{i}{\hbar}\underbrace{\int_0^{T} H_1(t)\,dt}_{>0}\Bigg]_> ,$$

where we have standard time ordering on the right and the reverse order on the left. In view of the positivity of both $H_1(t)$ and $H_2(t)$ this can only by realized under exceptional circumstances. The Humpty-Dumpty problem is then: For which $\varrho(0)$ and $H_1(t)$ is it possible at all? There are, of course, two very important rules in this game, namely (i) only real physical interactions are considered, and (ii) over-idealizations are not allowed.

4. Spin Coherence in a Stern-Gerlach Interferometer

Maybe part of the evolutionary changes can be undone at least? Consistent with rule (i) we ask more specifically: Can the two partial beams of a Stern-Gerlach apparatus (SGA) be reunited with such precision that the initial spin state is recovered? This problem of constructing a SG interferometer has been studied some time ago by Schwinger, Scully, and myself [1, 2]; a gedanken experiment based on a functioning device was also analyzed [3]. It is found that one needs three more SGAs for the beam reunion. Ideally, the four SGAs would be perfectly identical. With due respect to rule (ii), however, we have to ask how large a mismatch between the SGAs can be tolerated. Two relevant quantities are the net transverse momentum transferred to either partial beam and their net transverse displacement, measured by

$$\Delta p = \int dt\, F(t)\,, \quad \Delta z = -\int dt\, F(t)\, t/m\,,$$

where $F(t)$ is the force on the up-component, for instance, produced by the inhomogeneity of the magnetic field. Thus, Δp and Δz are properties of the macroscopic SGAs; the deviations from the ideal values $\Delta p = 0$, $\Delta z = 0$ are resulting from the lack of control over the SGAs. It turns out that to maintain spin coherence one must, at least, ensure that

$$(\delta z\, \Delta p/\hbar)^2 + (\delta p\, \Delta z/\hbar)^2 \ll 1$$

holds, where δz and δp are the natural spreads of the beam prior to entering the first SGA. Therefore, one can tolerate only such variations in the magnetic field that both $|\Delta z| \ll \hbar/\delta p$ and $|\Delta p| \ll \hbar/\delta z$, with the consequence

$$\frac{|\Delta z|\,|\Delta p|}{\hbar} \ll \frac{\hbar}{\delta z\, \delta p}\,.$$

In summary, this states that the *macro*scopic magnetic field must be controlled with *sub-micro*scopic precision. It won't be easy.

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* If you don't have a copy of Mother Goose at home, you can look up the Humpty-Dumpty riddle in [2] and [3].

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